

Synthetical a priori, schematism, and diagrammatical practices

For several decades, the relevance of the Kantian notion of “synthetical a priori” for our understanding of scientific and mathematical activities, has been, if not exactly re-evaluated, certainly re-vindicated against the verdict, common in the early 20th century, that such a notion had become obsolete after the transformation and pluralization of geometrical and physical scientific theories.

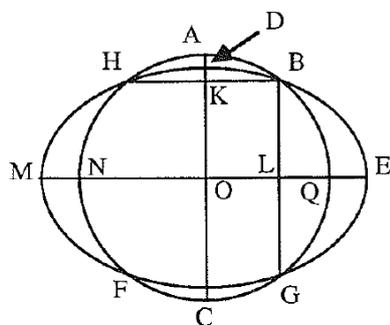
I would like, with this paper, to contribute to this enterprise, by revisiting several Kantian themes and texts in the light of *another* recent re-evaluation in history and philosophy of mathematical practice: that of diagrammatical practices in geometry. This re-evaluation has been twofold, asserting, first, that diagrammatical practice did *not* rely on the *representation* of concepts in drawn figures, and, second, that it was in a sense *still at play*, including in contemporary *formal* mathematics. I will try to link these two re-evaluations by locating, in Kant, a *confusion* between *two different facets* of a problem, that is, two distinct meanings of what Kant calls “schematism”, as far as mathematics are concerned.

I. Reevaluating diagrams

Concerning diagrams, and more generally the use of spatial and temporal intuition in mathematics, the basic idea is that there had classically been two joint mistakes about them: the first was to believe that only classical geometry, or more generally non-formal mathematics, use spatial reasoning through diagram manipulation; the second was to believe that classical geometry, or diagrammatical techniques in general, rely on the *representation* of some properties in some figures. Both mistakes are historically directly correlated, and we have been told the story that “Euclid was not rigorous because he relied on the (representational) quality of his diagrams”, and that in return “we do have rigor after Hilbert by avoiding to rely on spatial intuition”.

a. A diagram is not a well-drawn picture

There are many ways to show why this perspective is wrong. A key point, highlighted for example by Ken Manders, is that diagrams in Euclid could not possibly work, in general, because of their representational quality, because they play the same inferential role in the very numerous *reductiones ad absurdum* that we find in the *Elements*, where what is “represented” is definitionally an *impossible* figure (the diagram of which cannot therefore be geometrically *representative*).



Euclid, *Elements*, III, 10:

Two circles impossibly crossing themselves in more than 2 points, rendering possible to diagrammatically prove that O, impossibly, is the center of two different crossing circles.

The reason this point is meaningful, is that a diagram is an inferential tool, allowing us to draw conclusions in a regulated manner according to what can be drawn and seen in the diagram. For us to be able to still do this in the case of an impossible figure, it is necessary that the reason for our

inferential success not be the representational accuracy of our picture! More generally, the very idea of representational role is problematic because, strictly speaking, as is well known, geometrical entities are *unpresentable*. It is, for starters, impossible to *draw* a line having length without width, or a point having no magnitude.

b. Diagrams as material, imaginative, and intuitive anchors

The real use of diagrams, as it has been studied for a few decades by psychologists, anthropologists, historian of mathematics and mathematician themselves, but also as it had already been theorized, according to David Rabouin, by Proclus, consists in producing, in spatial imagination, not an entity that has the properties of its concept, but a configuration such that our imaginative action parallels or equals the properties of the concept. For example, a point in a diagram is not the representation of an adimensional being, but the production of an element in the imagination such that, in it, our imaginative division stops, in the same way that the point is conceptually thought as that in which no more division is possible.

Once we have (thanks to Proclus) such an understanding of Euclid, it is possible to re-apply this understanding to *all* mathematical techniques, those of amateurs and professionals alike, as much those of the children who count on their fingers, as those of the formalist who operates with lines on paper or on a screen. In all cases, even those not strictly diagrammatical, the inferential process is *anchored* in a visualization and a manipulation that must follow a set of a priori given rules. In other words, they rely on what Kant called “intuition”.

| Step | Reason |
|---------------------------|---------------------------------|
| 1. $\neg p \wedge q$ | Hypothesis |
| 2. $\neg p$ | Simplification |
| 3. $r \rightarrow p$ | Hypothesis |
| 4. $\neg r$ | Modus Tollens using (2) and (3) |
| 5. $\neg r \rightarrow s$ | Hypothesis |
| 6. s | Modus Ponens using (4) and (5) |
| 7. $s \rightarrow t$ | Hypothesis |
| 8. t | Modus Ponens Using (6) and (7) |

Producing truth through the intuitive application to formal symbols of a priori given inference rules.

II. The two facets of schematism

What does this have to do with Kant’s schematism? It allows us to get a better grasp on a duality at the heart of Kant’s theory. We know that, in Kant’s *Critique of Pure Reason*, “schematism” is the name of the operation of the imagination which allows us to apply concepts to sensible intuition, and that such an operation plays a crucial role in mathematical knowledge – which proceeds, from schemes, by way of “concept construction”. But Kant himself, especially in his brief and remarkable comments on algebra, shows that there are in fact *two distinct facets* of schematism, as far as mathematics are concerned; that is to say a facet we could call “representational”, and one we would call “diagrammatical”.

a. Representational or ontological schematism: *passé*

The first facet of the scheme is the one most explicitly developed by Kant, and its purpose is indeed to indicate us how a concept might be able to apply to an object in “reality” or experience. The reason we need “schematism”, in this first sense, is because we need to ensure that mathematical concepts

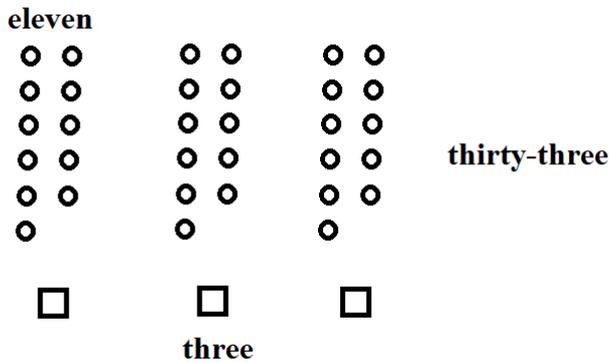
can be *realized*, realization that for Kant takes the representational form of an “image”. At this first level, we can say that for Kant the pure intuitions of space and time warrant the *a priori applicability* of number, and of Euclidian geometrical concepts and constructions; they ensure that a mathematical concept can find a place in a reality that is still understood as *representable*. And in this way, schemes serve in mathematics the same function they serve in general in the system, that of ensuring a place for concepts in the domain of sensible representation. In *this* sense, and as a result of the well-known evolution of contemporary physics and mathematics, schematism might indeed seem *passé*: we do not think that some spatial intuition is able to a priori guaranty the applicability of Euclidian geometry to the physical universe, or that we know mathematical truths because of their necessary realization in experience.

b. The independence of diagrammatic schematism

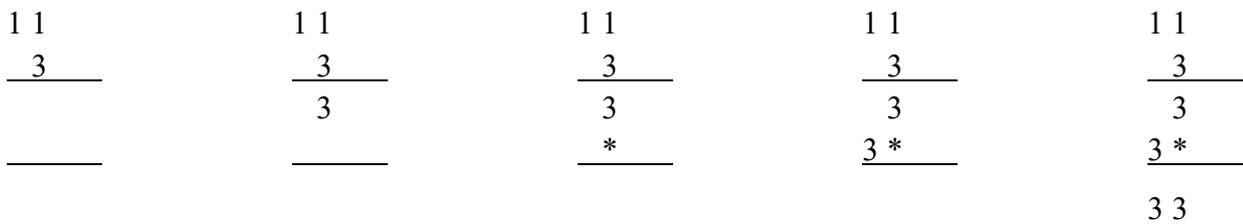
However, the scheme also plays another, more specific, role in mathematics, and indeed a rather different one: that of allowing the uniquely mathematical practice of “concept-construction”. Let us follow Kant’s presentation in the “Methodology”, as it was assessed in (Salanskis1991). To “construct” a concept is to “*a priori* represent the intuition that corresponds to it”, and, in principle, such a task falls to schematism. But if, usually, schematism does not produce a *singular* intuition, that is to say an *image* – but only a general rule that gives the *form* of the spatio-temporal effectuation of the relations contained in the concept: the anticipation of what will count as the form of a dog, the anticipation of the temporal form of causality in general, etc. – such is *not* the case where concept-construction is concerned. There, schematism allows us to *singularly construct*, in the intuition, what is universal in the concept. For example, we can construct a singular, drawn, triangle, *seen as* a generic triangle. Moreover, schematism works so as to make us *see* in the singular intuition some properties that *then* fall back to the concept itself, widening our conceptual (synthetical a priori) knowledge, in a reverse process which is a sort of “counter-schematism” and which is a defining trait of mathematical knowledge for Kant.

But I do not, in fact, need to have the “scheme” of a concept in the first, representational sense of the term, nor the capacity to produce, in a strict sense, what Kant calls its “image”, to be able to mathematically manipulate this concept. This is fortunate, because, as Kant knows perfectly well, it is in general completely impossible to produce or visualize an *image* of mathematical objects, even at the very elementary level of a somewhat great integer, such as 100 000 – although the general scheme of number is supposed to ensure us of its “realizability” in space and time. This impossibility does not, however, prevent me to mathematically *manipulate* this object, by way of a regular *construction*, and to deduce some knowledge in return about it. How so? By operating not directly on an image of its pure multiplicity of 100 000, but on this sort of “image” that is symbolic writing, in other words by manipulating its *digits*.

Let us consider for example the basic operation $11 \times 3 = 33$. I *might* be able to visualize a multiplicity of 11, and to visualize its triplication, and the result of 33, at the level of spatially distributed multiplicities embodying as *images* the “schemes” of those numbers in the first meaning of the term.



But this is not *in fact* what I do in primary school when I learn $11 \cdot 3$. In school, I learn to manipulate the *digits* according to the rules of multiplication (“three times one, three, I write down “3” and go down a line...”). And I can know (reflecting on it, later in life) that the process is valid, because I can a priori know that the process of representing integral multiplicities in positional decimal writing, operating on the digits, and re-interpreting the decimal result in terms of integral multiplicities, preserves truth and applicability.



In this second sense of schematism, I still possess, in some way, a scheme of the concept, inasmuch as I have a rule telling me how to produce a certain image, and how to get new conceptual properties out of it. But I do not strictly speaking produce an image *of* the concept, in the sense of the first schematism. I do not have a rule telling me how to effectively *realize the object of the concept in representation*, only a rule telling me how to *produce a (symbolical) image of it that can be manipulated* and that allows us to deduce some truths about it.

c. Schemes, diagrams, and a priori intuition everywhere

Now, our detour through the history and philosophy of mathematical practice might lead us at this point to an interesting thesis: that it is, in fact, and contrary to what Kant himself seems to suggest, *always* this *second* mode of schematism that is effectively used in mathematics, in elementary arithmetic as well as in Ancient geometry, and contemporary practices. If this is true, then it follows that diagrammatical activity is, in general, freed from the question of the representability or “reality” of the objects considered. It remains possible to “schematize” a mathematical concept, in the second sense, quite independently from the question of whether the concept can or cannot have a scheme in the first sense of the term. Therefore, schematism in the second sense is retained even in the case of “ideal”, “abstract” or “unpresentable” mathematics.

Conversely of course, the key point – which is remarkably already raised by Kant himself in his comments about algebra – is that *symbolic manipulation still belongs to the domain of concept-construction in space and time*. Indeed, as “absurd” or “abstract” can its results be, mathematics is always anchored in the possibility of such *a priori* spatio-temporal operations as are the manipulations

of formal or algebraic symbols. In this sense, its activity remains diagrammatical, and grounded on an *a priori* synthetical knowledge about the symbolical operations themselves.

III. Schematism and formal systems

Let us finally consider a particular role of schematism, pertaining to the relationship between mathematical practice and formal axiomatic systems. We talked, about the trivial case of elementary arithmetic, of the process of justification by way of conservativity from one symbolic stratum to the next; it is remarkable that, like diagrammatical schematism, this process is independent from the possibility of a “realization” by representational schematism, and that it too is omnipresent and indispensable to practice. This is manifested in formalism: by making absolute and exhaustive explicitation a possibility, formalism guaranteed a unprecedented kind of rigor, but at the same time immediately forced us to find a way not to have to use the fundamental formal symbols themselves, those used at the level of complete explicitation, save one is forced to use several thousand signs to express the most elementary of concepts. One never actually uses the formal system, but in formal mathematics, one, ideally, makes sure that it would be *de jure* possible to formulate the theory in the system. We in fact have a *scheme* that ensures the possibility of realizing a higher-order symbolism in a lower-order one, and vice versa.

The question I want to ask here is then the following: how are we to understand schematism in the context of the relationship between actual mathematical practice and foundational formal systems? And the answer seems to be, that it actually depends of what sense or facet of schematism is here considered!

a. The Formal system as the “Real” of formal mathematics

As far as the first, representational, facet of schematism is concerned, we might say that curiously, the symbolism of the formal system itself plays, in a way, the role played by the “real” in Kant’s theory: the metamathematical instructions tell us what its possible elements (the symbols) are, and what combinations of those elements form possible entities (the *well-formed formulas* being the “things” of this peculiar “real”), and what types of “motion” are possible or necessary (the inference rules such as *modus ponens*). The *theorems* are then analogous to the “existing beings” (the true statements) of this world, to be distinguished from its merely *possible* things (the well-formed formulas), and among which the *axioms* are counted from the start. *Axiom schemata* are *a priori* rules that determine the “existence” of certain “beings” of this world (that is to say the *a priori* truth of certain statements; this pseudo-“existence” is not to be confused with the existential import given *inside the formal system* by some of the axioms). The study of the theorems of the formal system is then analogous to the “physics” of this “world”, or the “geometry” of this “space”.

In a version of ZFC that only admits of set membership and equality as non-logical operative symbols, already *inclusion* becomes a concept in need of a scheme, in the sense that the symbolic writing “ $x \subset y$ ” needs to be able to be “represented” in the “real” of the system, that we have a rule to produce an “image” of it, that is a formula of the form “ $\forall z z \in x \Rightarrow z \in y$ ”. Mathematical constants such as the symbol for the null set need to be able to be interpreted in diverse ways according to the place they have in a statement, etc.

b. The opposite directions of formal schematism

In the one hand, therefore, the formal system could be seen as the “*domain of representation*”, by

projection into which one could “embody” higher-order concepts (e.g. ordinals, lattices...), via some sort of a “schematism” in the first sense, and through which one could, in reverse and via a form of “counter-schematism”, learn some new truths about them; this presentation seems not entirely out of place, considering that the “truth” and “validity” of practical mathematical statements is ideally supposed to be guaranteed by the ur-truth and ur-validity of the axioms and the fundamental formal rules. And in this sense, schematism would operate in a *top-down* manner, from higher-order concepts to the lowest level of symbolic language.

But, on the other hand, what is officially and according to Kant supposed to be the proper working of schematism in mathematical concept-construction, that is to say the projection of concepts in a space of practical symbolic manipulation, this takes place in an opposite bottom-up manner! The schematic projection, the figuration conducive to manipulation, this happens at the highest level possible, the relatively informal level, on paper, where operations are relatively clear and easy. This space of presentation and manipulation is, to be noted, still a *space* and a *time* of a *priori* figuration, an *intuitive* space and time in which I write and I draw.

In this sense, a concept which, in theory, should acquire its ultimate (*absence of*) meaning in the official formal language, also needs, if mathematics is to actually be done, a scheme *in the second sense of the term*, in the direction of a higher-order representation, which no longer needs to be rigorously and sternly formalistic, because it is not a “real”, but an *imaginative tool*, although it necessarily is 1/ embodied in space, and 2/ subjected to a priori rules.

Appendix:

Kant, *Critique of Pure Reason*, II. Transcendental Doctrine of Method, Ch. 1: The Discipline of Pure Reason, sc. 1: The discipline of pure reason in dogmatic use. A716-8/B744-6.

Trans. and ed. by P. Guyer & A.W. Wood, The Cambridge Edition of The Works of Immanuel Kant, 1998, pp. 631-3.

“Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.

But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitatem*), as **in algebra**, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of

magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc. and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts.

What might be the cause of the very different situations in which these two reasoners find themselves, one of whom makes his way in accordance with concepts, the other in accordance with intuitions that he exhibits a priori for the concepts? According to the transcendental fundamental doctrine expounded above, this cause is clear. At issue here are not analytic propositions, which can be generated through mere analysis of concepts (here the philosopher would without doubt have the advantage over his rival), but synthetic ones, and indeed ones that are to be cognized a priori. For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. Now this is not possible in any way but by determining my object in accordance with the conditions of either empirical or pure intuition. The former would yield only an empirical proposition (through measurement of its angles), which would contain no universality, let alone necessity, and propositions of this sort are not under discussion here. The second procedure, however, is that of mathematical and here indeed of geometrical construction, by means of which I put together in a pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general synthetic propositions must be constructed.”

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